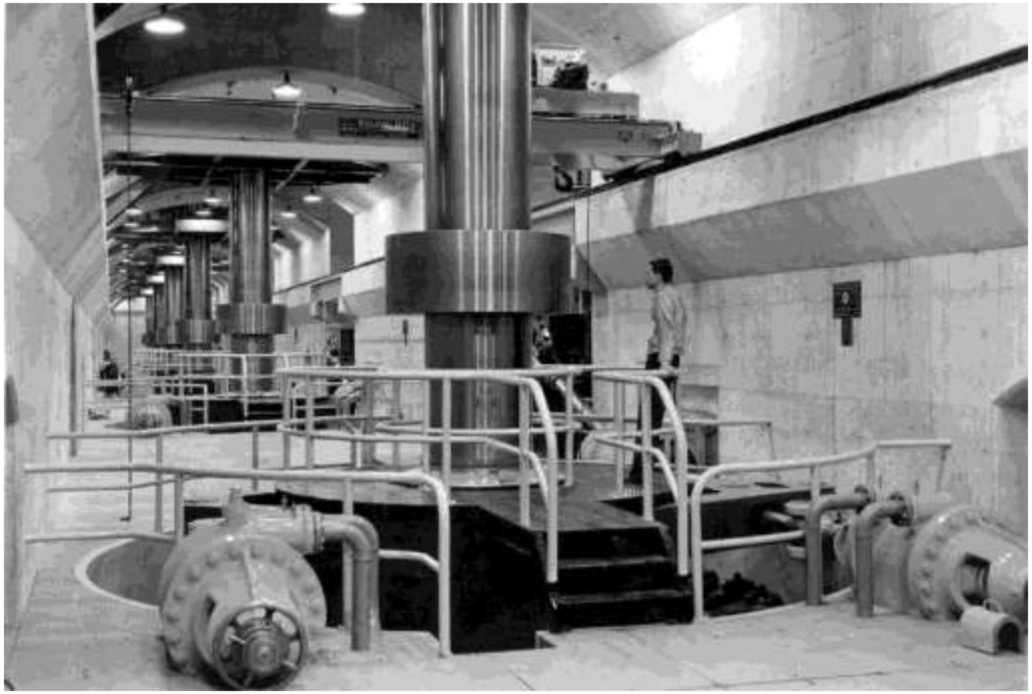
Chapter 3

Torsion

Torsional Loading

This part is devoted to the study of torsion and of the stresses and deformations it causes. In the hydroelectric plant shown , turbines exert torques on the vertical shafts that turn the rotors of electric generators.

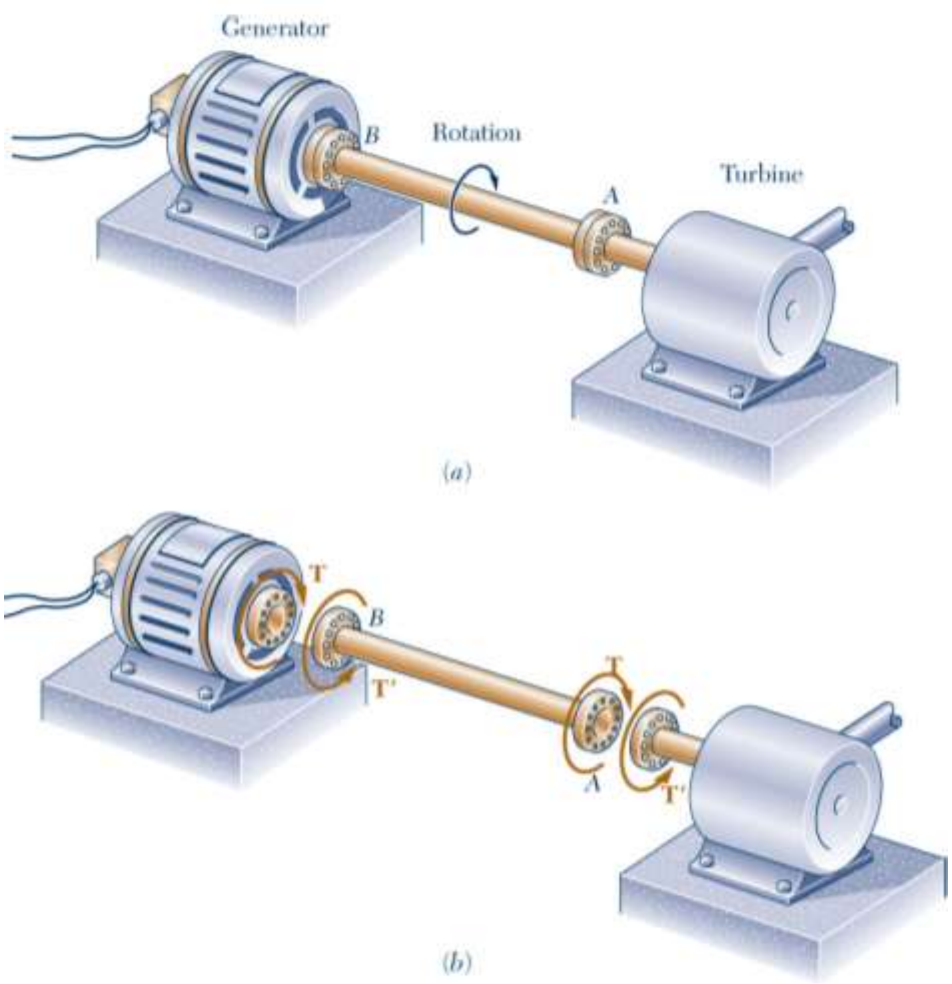
Structural members and machine parts that are in **torsion** will be considered



For a circular shaft subjected to a given torque, we will determine the distribution of **shearing stresses**, find the **angle of twist**

We will account for stress concentrations where an abrupt change in diameter of the shaft occurs.

Torsional Loads on Circular Shafts



(a)

(b)

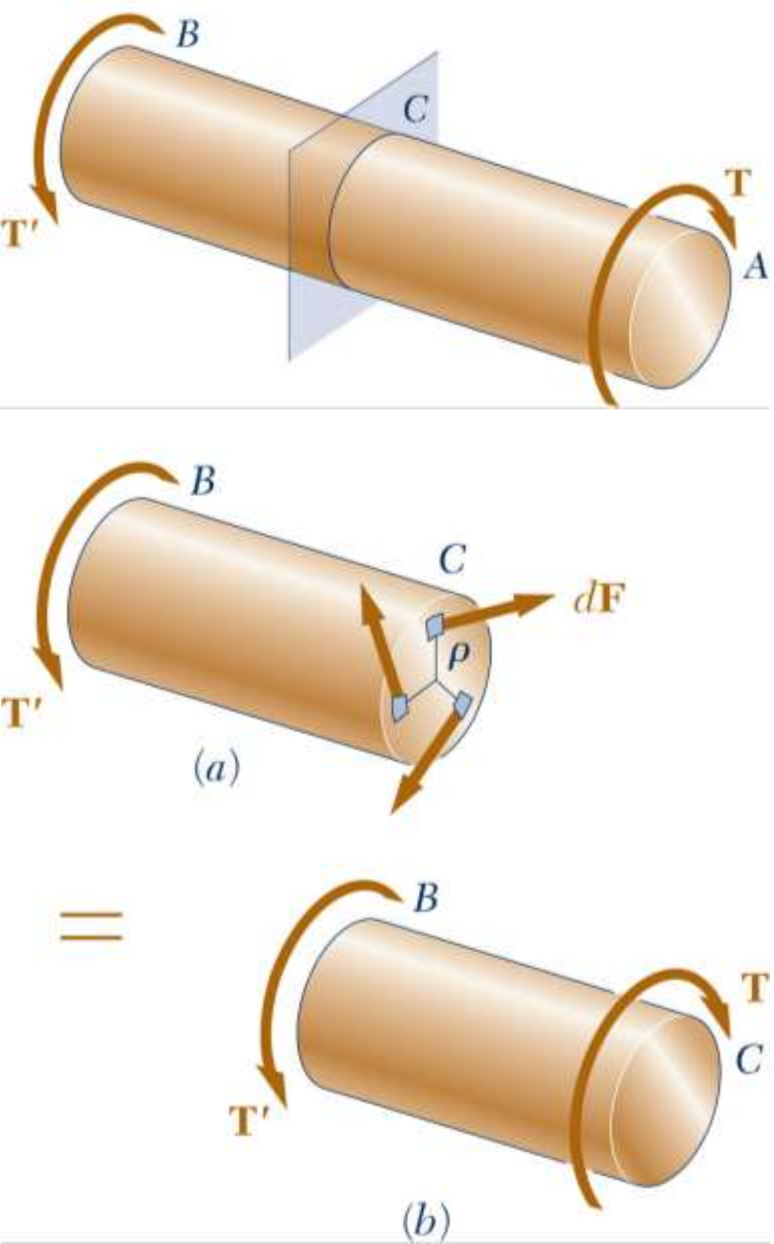
Interested in stresses and strains of circular shafts subjected to twisting couples or **torques**

Turbine exerts torque T on the shaft

Shaft transmits the torque to the generator

Generator creates an equal and opposite torque T'

Net Torque Due to Internal Stresses



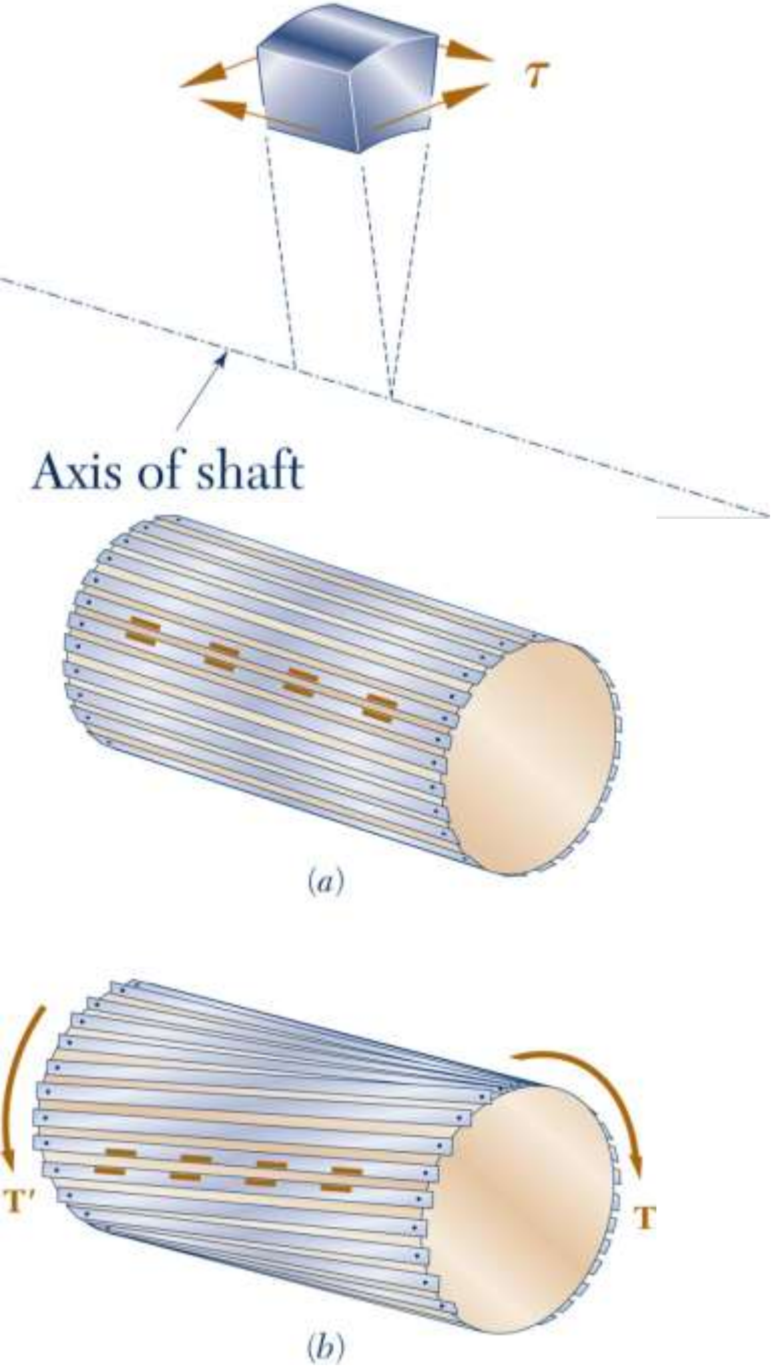
Consider a shaft subjected to equal and opposite torques at its opposite ends. The net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

Although the net torque due to the shearing stresses is known, the distribution of the stresses is not. The distribution of shearing stresses is **statically indeterminate**. (must consider shaft deformations)

Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads **can't be assumed uniform**.

Axial Shear Components



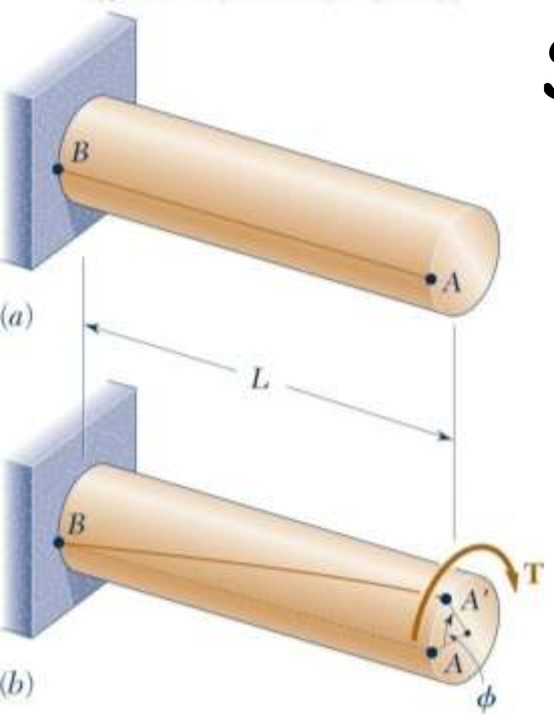
Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.

Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft.

The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

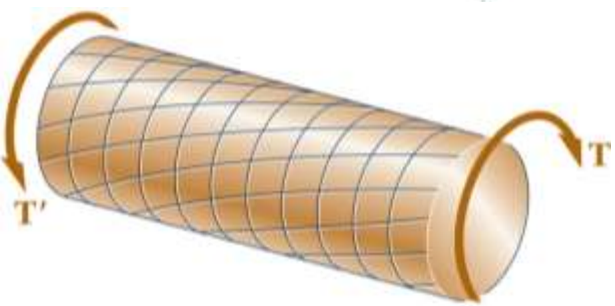
The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations



From observation, the angle of twist **Φ** of the shaft is proportional to the applied torque **T** and to the shaft length **L**.

When subjected to torsion, every cross-section of a circular shaft remains **plane and undistorted**.

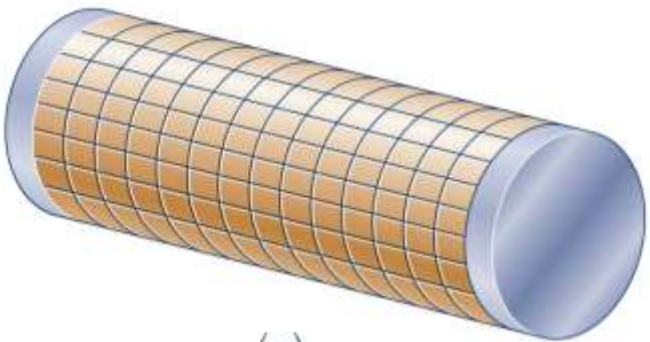


(a)

Cross-sections for hollow and solid circular shafts remain plain and undistorted because a **circular shaft is axisymmetric**.

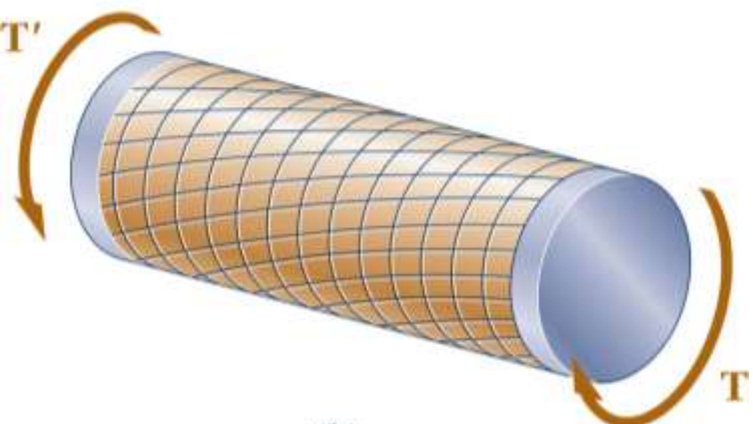
Cross-sections of **noncircular** (non-axisymmetric) shafts are **distorted** when subjected to torsion.

Ideal Torsional Model



(a)

We want to make sure that the couples are applied in such a way that the ends of the shaft remain plane and undistorted. This can be accomplished by applying the couples to rigid plats, which are solidly attached to the ends of the shaft.

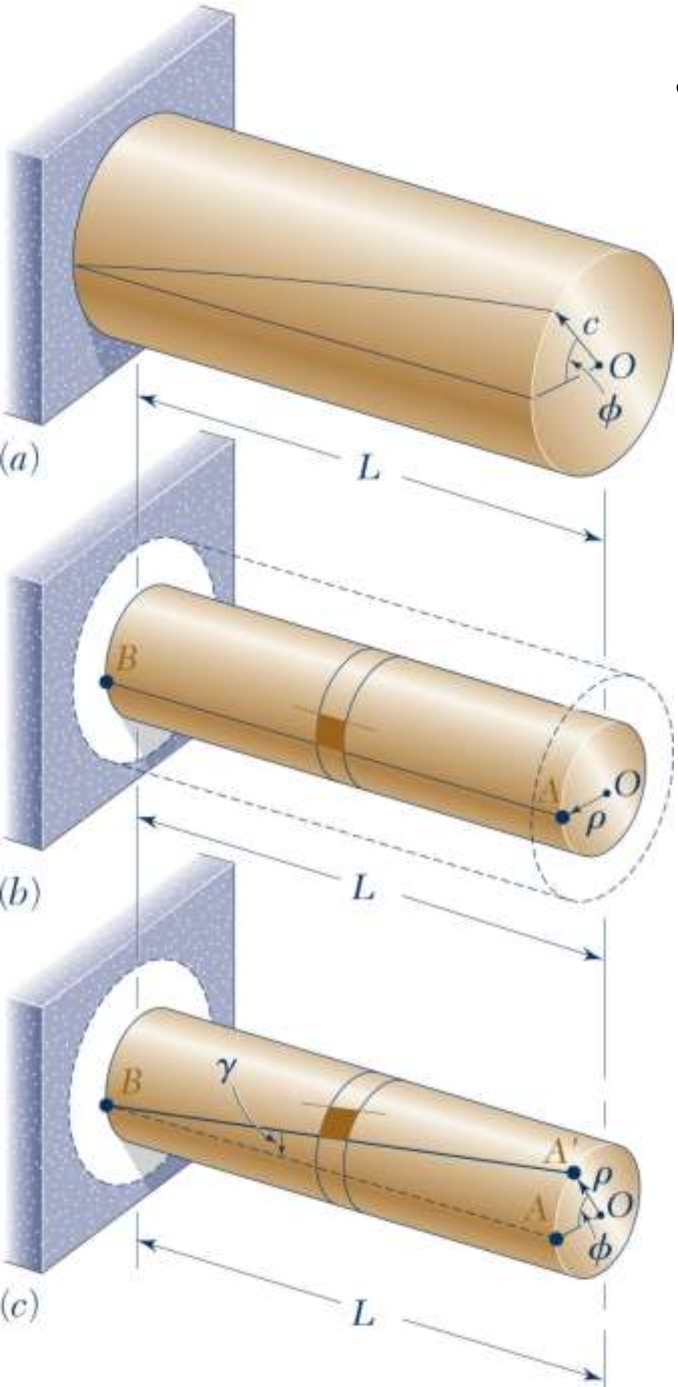


(b)

Loading conditions encountered may differ appreciably from those corresponding to this model

This model can be solved easily and accurately.

Shearing Strain



Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

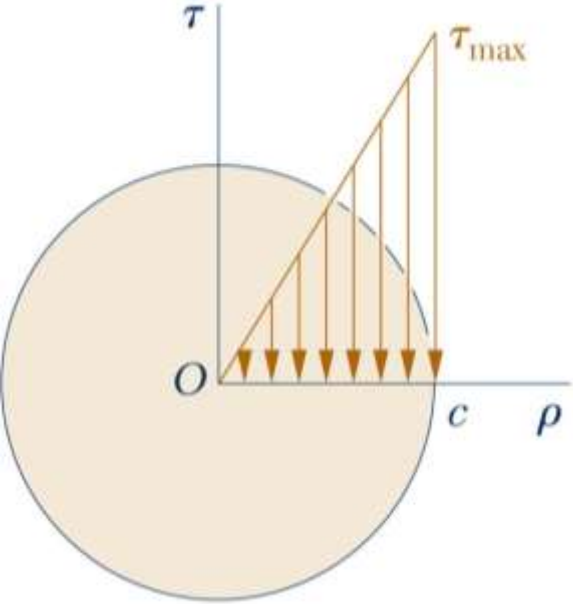
Since the ends of the element remain planar, the shear strain is equal to the angle between AB and A'B.

It follows that $L\gamma = \rho\phi$ or $\gamma = \dfrac{\rho\phi}{L}$

Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range

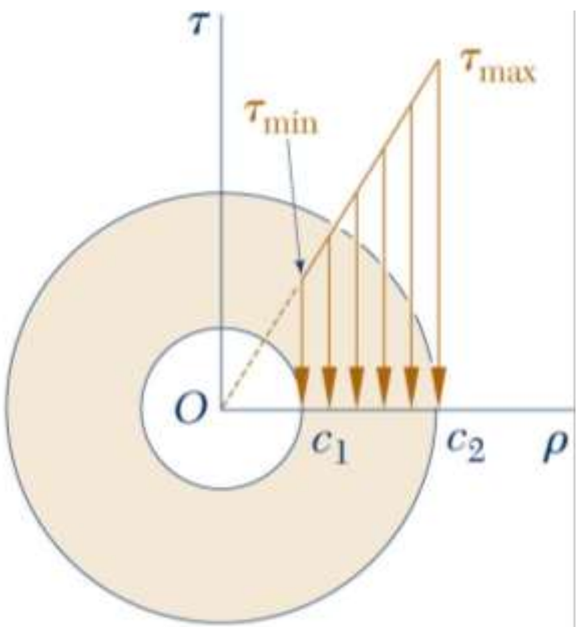


$$\gamma = \frac{\rho}{c}\gamma_{\max}$$

Multiplying the above equation by the shear modulus,

$$G\gamma = \frac{\rho}{c}G\gamma_{\max}$$

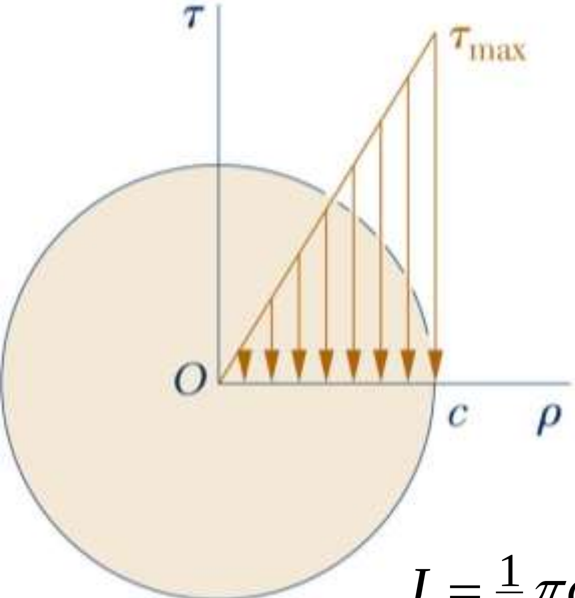
From Hooke's Law, $\tau = G\gamma$



$$\longrightarrow \quad \tau = \frac{\rho}{c}\tau_{\max}$$

The shearing stress **varies linearly** with the radial position in the section.

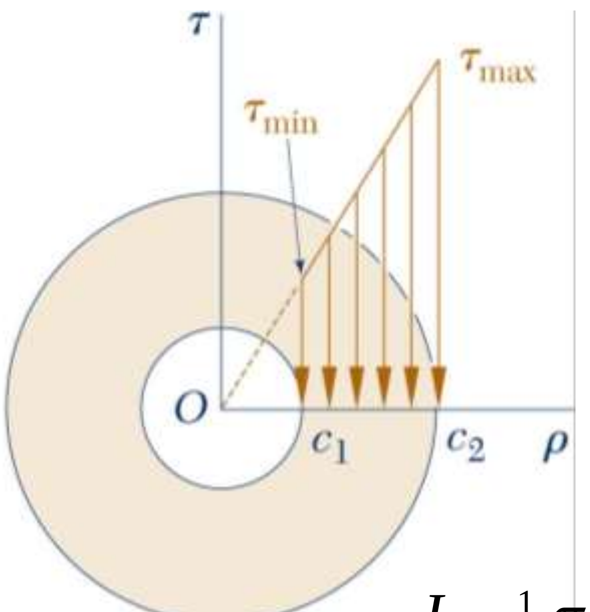
Stresses in Elastic Range



$$J = \frac{1}{2}\pi c^4$$

Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho\tau\, dA = \frac{\tau_{\max}}{c}\int \rho^2 dA = \frac{\tau_{\max}}{c} J$$



J is the polar moment of inertia

$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

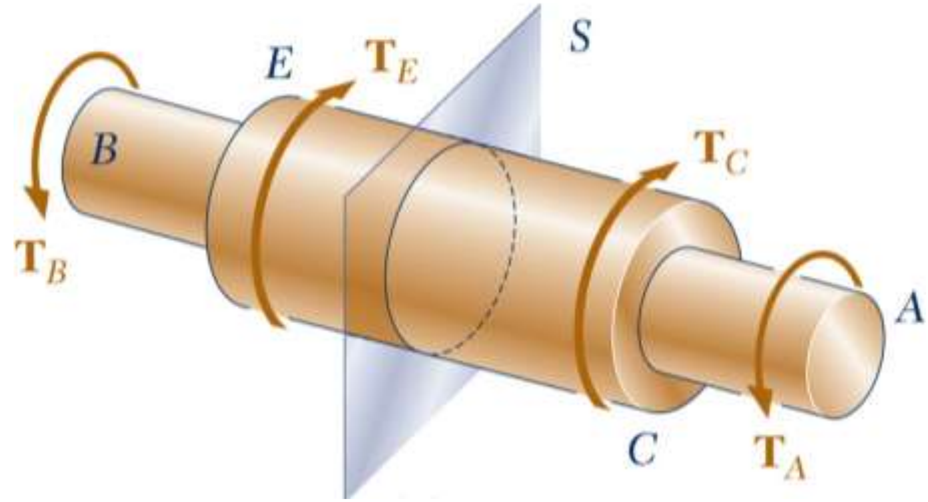
The results are known as the **elastic torsion formulas**,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\tau = \frac{T\rho}{J}$$

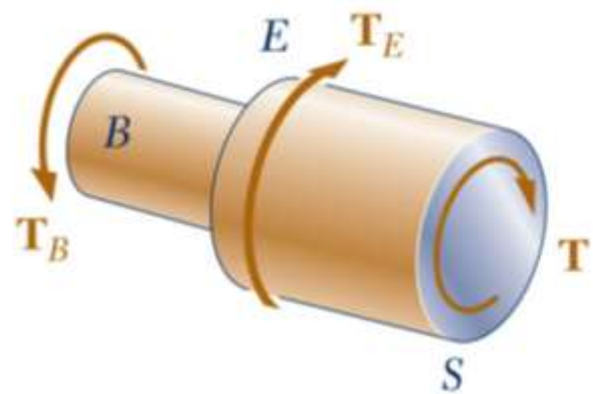
Stresses in Elastic Range

Consider a shaft with variable cross section and/or subjected to torques at locations other than its ends.



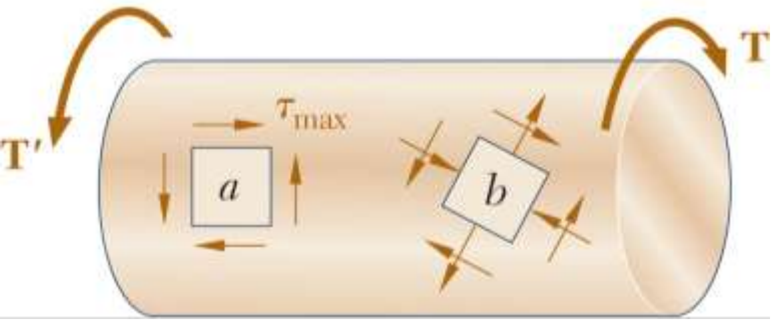
The distribution of shearing stress in a given cross section S of the shaft is obtained as

$$\tau = \frac{T\rho}{J}$$

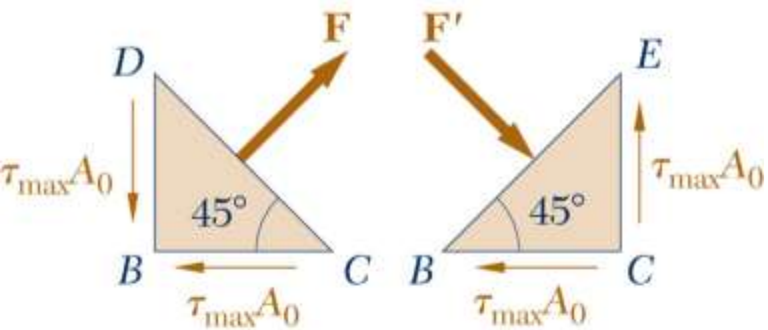


T is the internal torque in section S
J is the polar moment of inertia of S

Normal Stresses



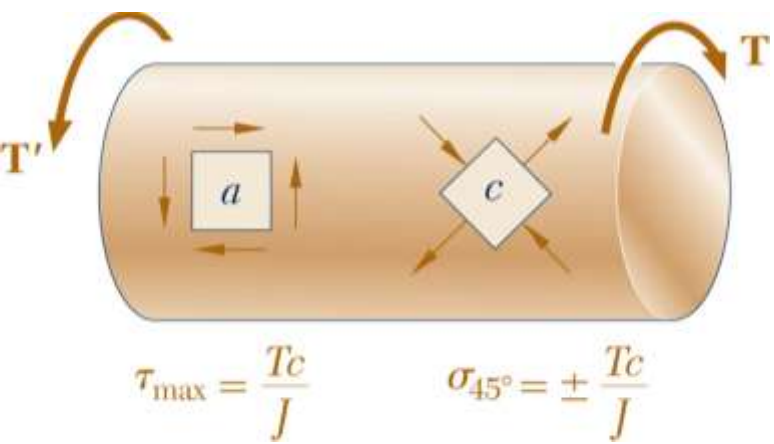
Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.



Consider an element at 45º to the shaft axis,

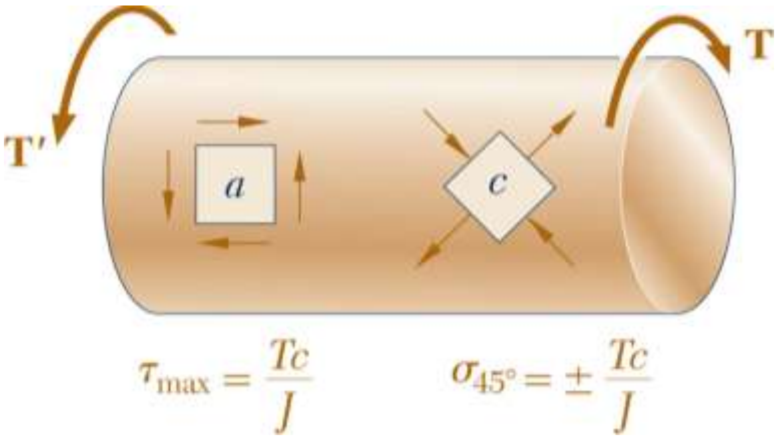
$$F = 2(\tau_{max} A_0)\cos 45° = \tau_{max} A_0 \sqrt{2}$$

$$\sigma_{45^o} = \frac{F}{A} = \frac{\tau_{max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{max}$$



Element a is in **pure shear**. Element c is subjected to a **tensile stress** on two faces and **compressive stress** on the other two. Note that all stresses for elements a and c have the same magnitude.

Torsional Failure Modes



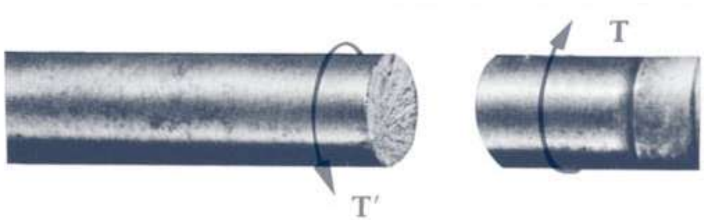
Ductile materials generally fail in **shear**. **Brittle materials** are weaker in **tension** than shear.

Ductile materials

When subjected to torsion, a ductile specimen **breaks** along a plane of **maximum shear**, i.e., a plane perpendicular to the shaft axis.

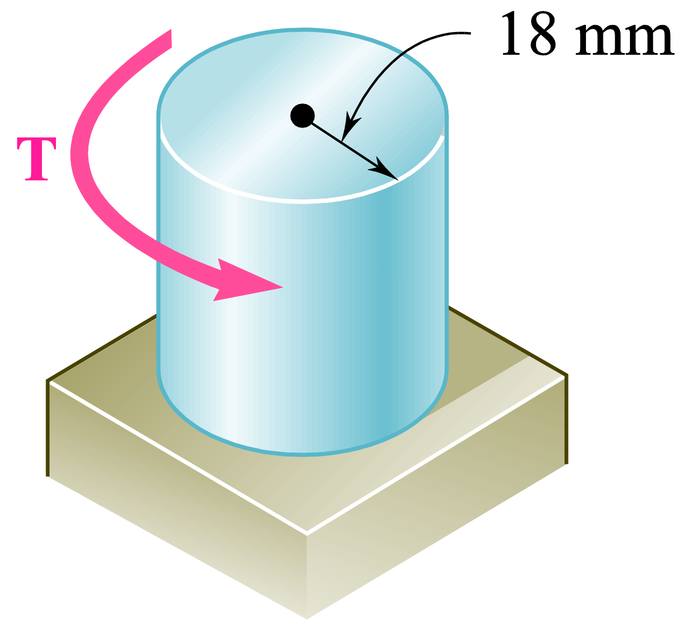
Brittle materials

When subjected to torsion, a brittle specimen **breaks** along planes perpendicular to the direction in which **tension is a maximum**, i.e., along surfaces at 45º to the shaft axis.



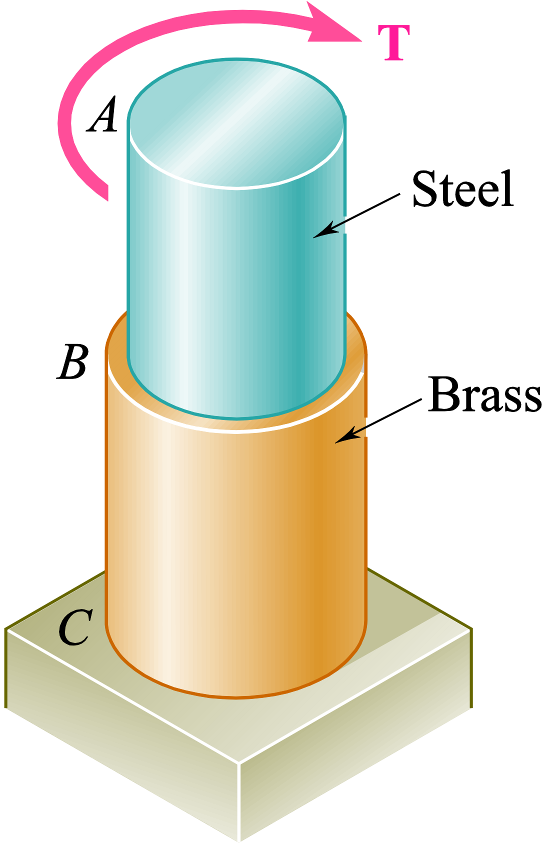
Example 1

3-2: For the cylindrical shaft shown, determine the maximum shearing stress caused by a torque of magnitude $T = 800$ N.m.



Example 2

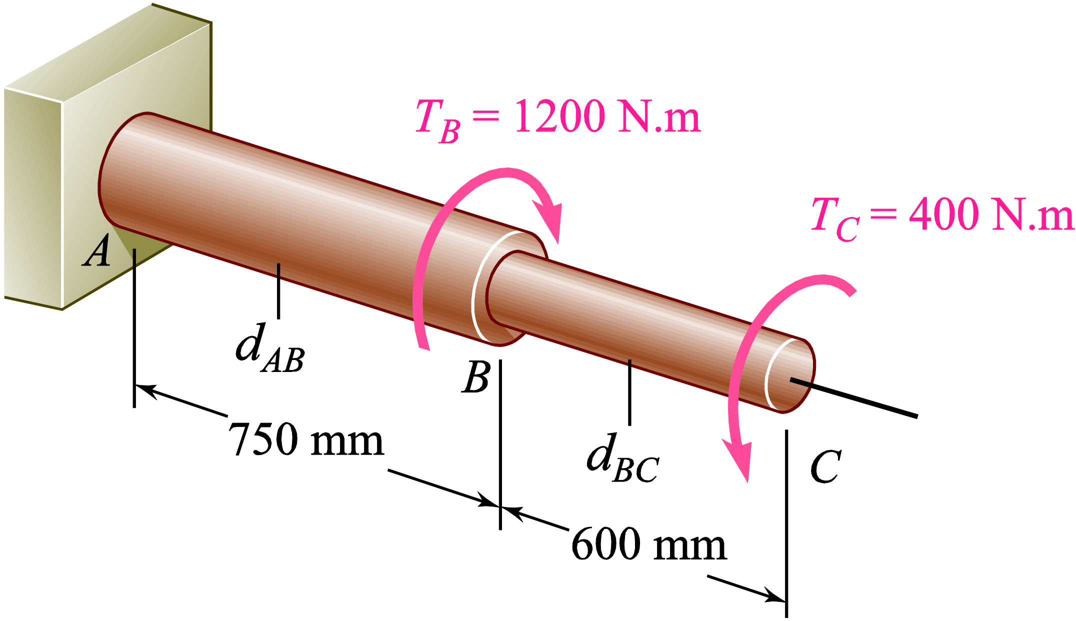
3-15: The allowable shearing stress is 100 MPa in the 36-mm-diameter steel rod AB and 60 MPa in the 40 mm diameter rod BC. Neglecting the effect of stress concentrations, determine the largest torque that can be applied at A.



Example 2

Example 3

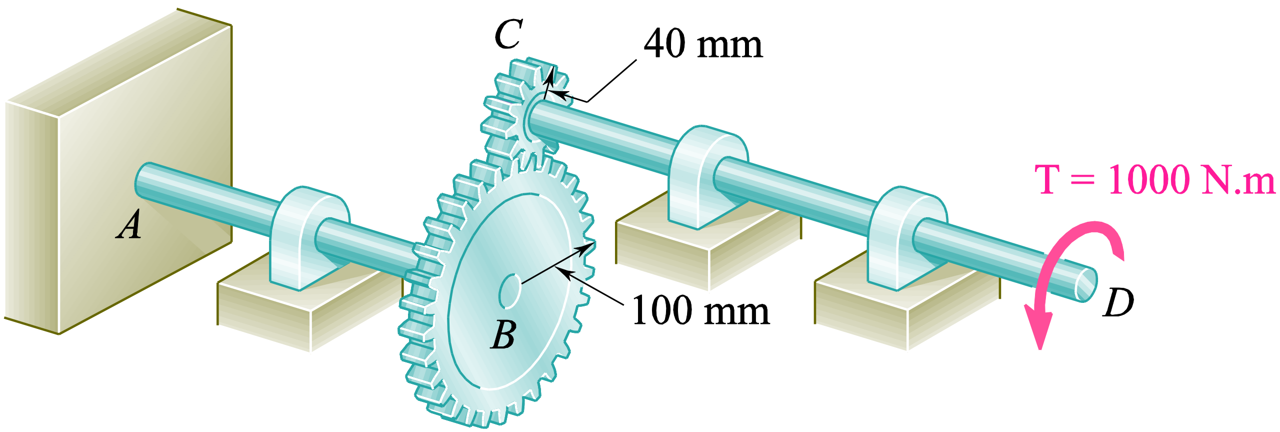
3-17: The solid shaft shown is formed of a brass for which the allowable shearing stress is 55 MPa. Neglecting the effect of stress concentrations, determine the smallest diameters d_{AB} and d_{BC} for which the allowable shearing stress is not exceeded.



Example 3

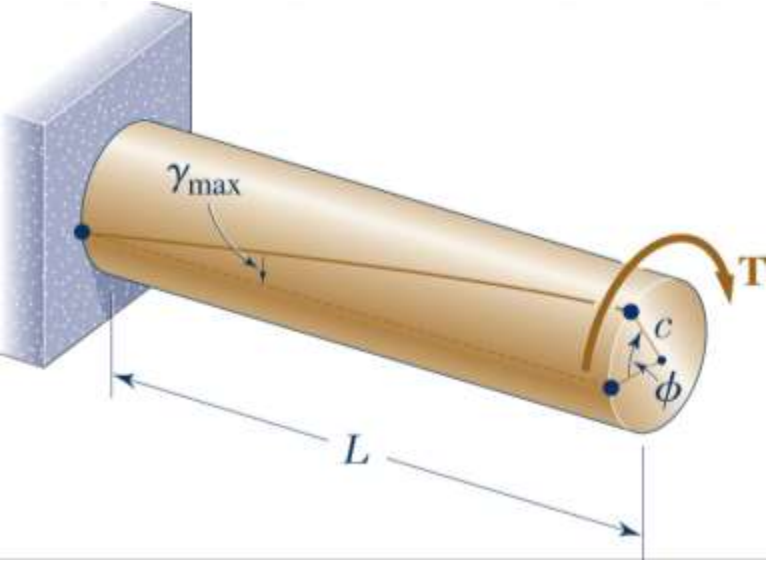
Example 4

3-21: A torque of magnitude T =1000 N.m is applied at D as shown. Knowing that the allowable shearing stress is 60 MPa in each shaft, determine the required diameter of **(a)** shaft AB, **(b)** shaft CD.



Example 4

Angle of Twist

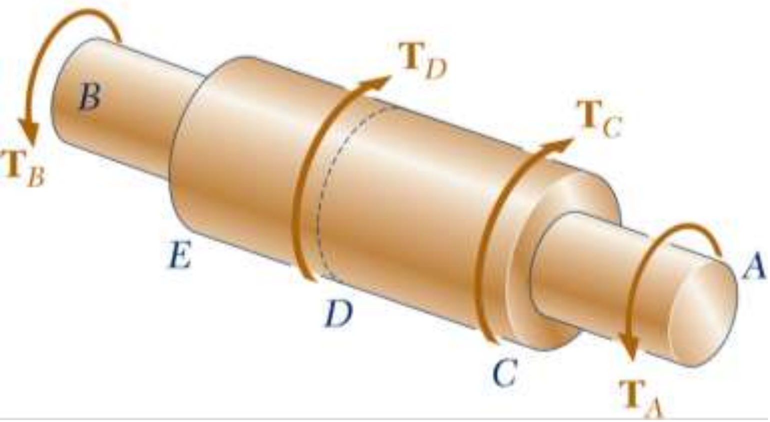


Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$



Equating the expressions for shearing strain and solving for the angle of twist,

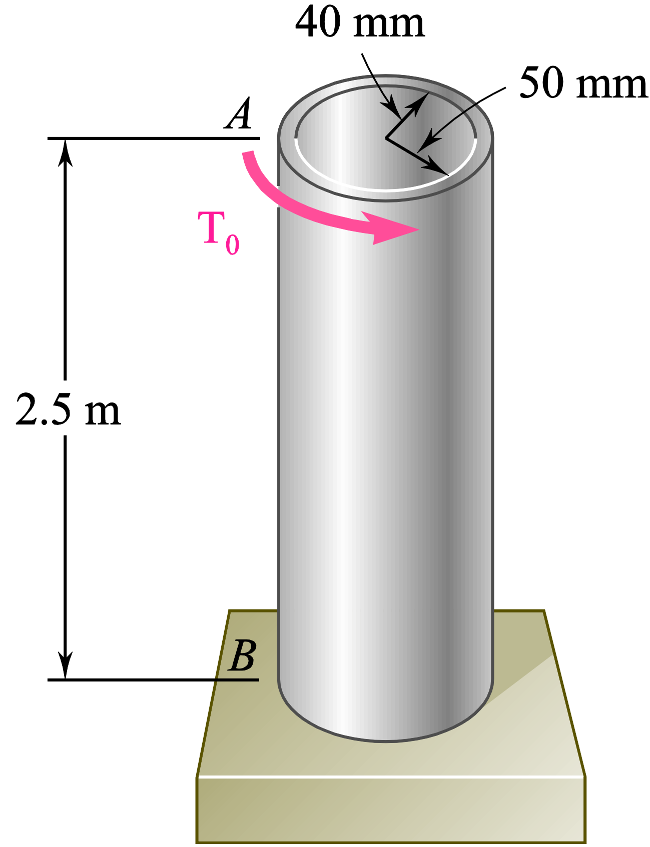
$$\phi = \frac{TL}{JG}$$

If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations.

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Example 5

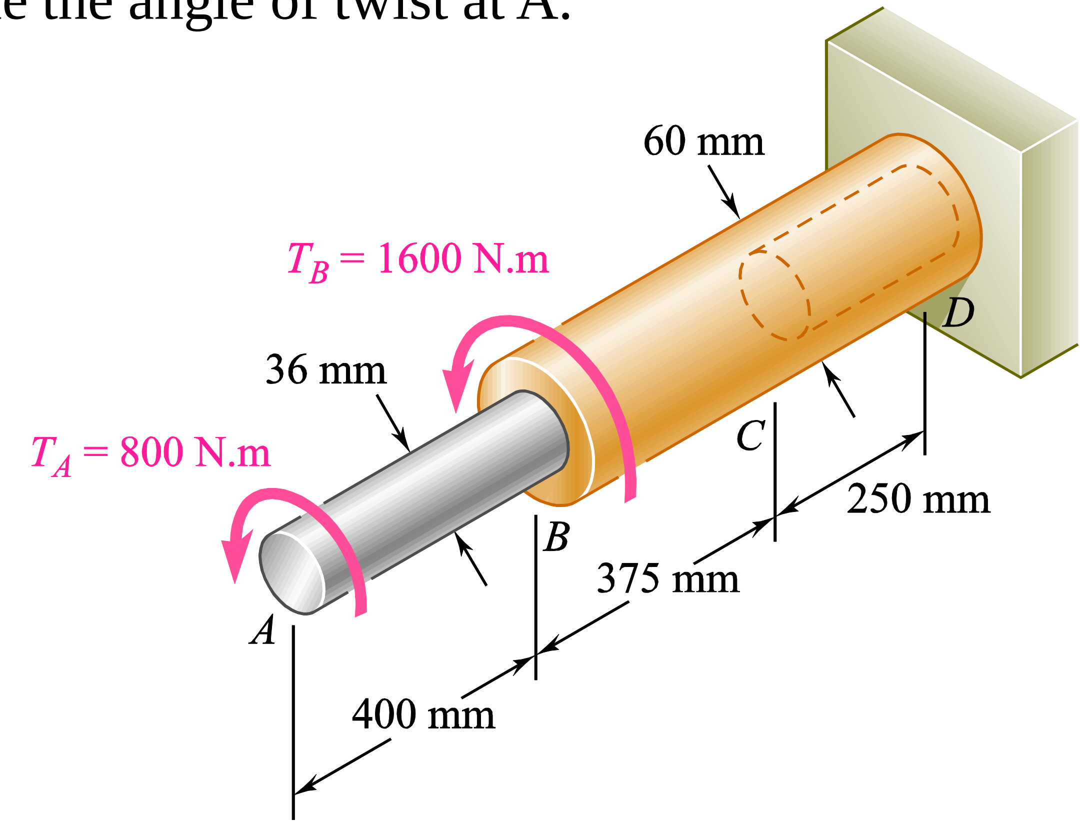
3-34: (a) For the aluminum pipe shown (G = 27 GPa), determine the torque $\boldsymbol{T_0}$ causing an angle of twist of 2°. **(b)** Determine the angle of twist if the same torque $\boldsymbol{T_0}$ is applied to a solid cylindrical shaft of the same length and cross-sectional area.



Example 5

Example 6

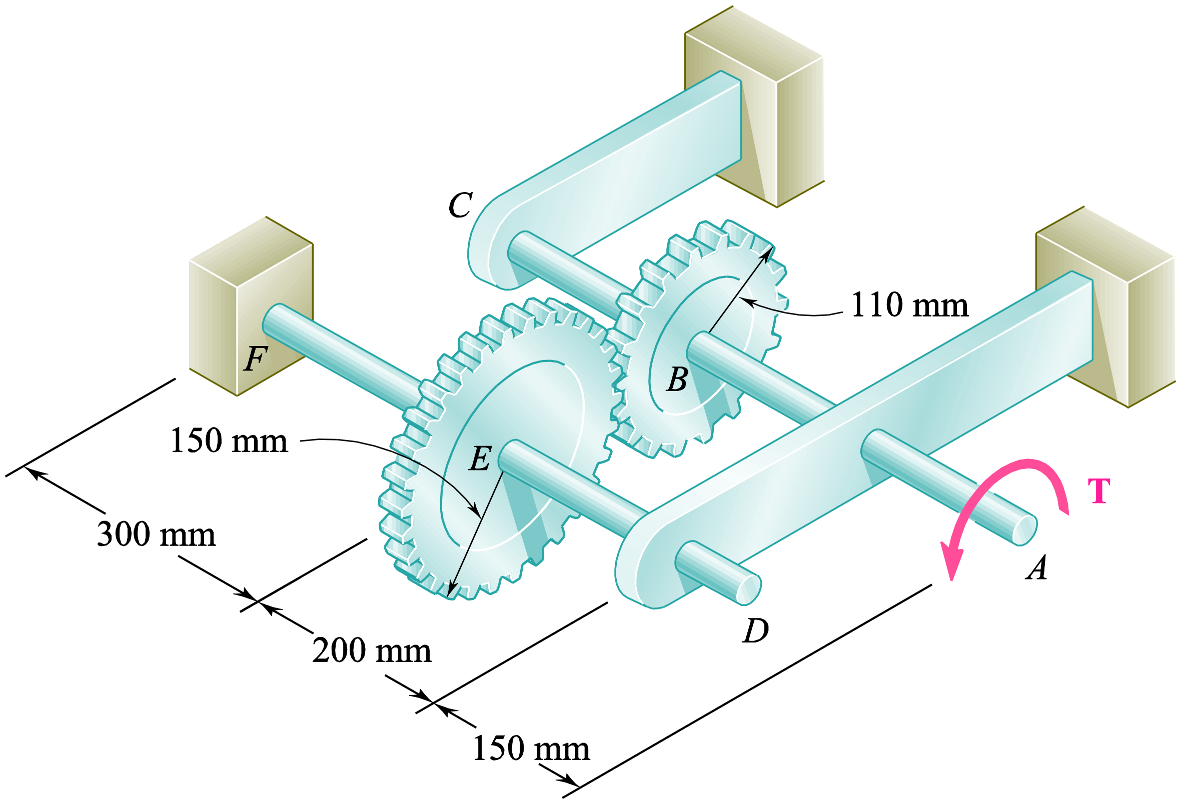
3-38: The aluminum rod AB (G = 27 GPa) is bonded to the brass rod BD (G = 39 GPa). Knowing that portion CD of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at A.



Example 6

Example 7

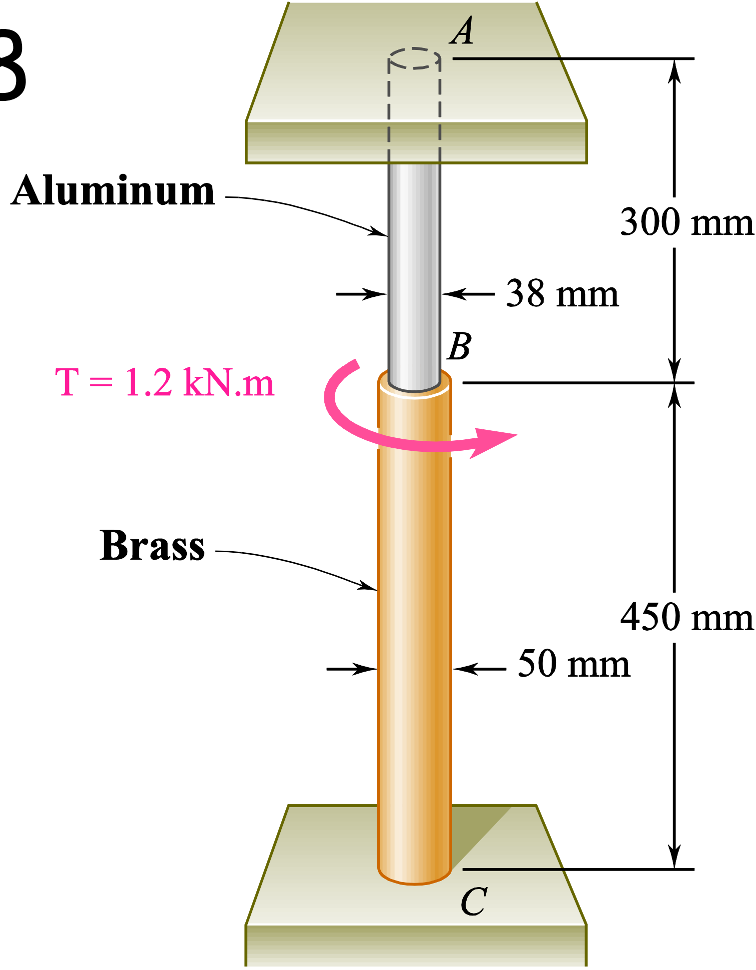
3-41: Two shafts, each of 22-mm diameter are connected by the gears shown. Knowing that G = 77 GPa and that the shaft at F is fixed, determine the angle through which end A rotates when a 130 N.m torque is applied at A.



Example 7

Example 8

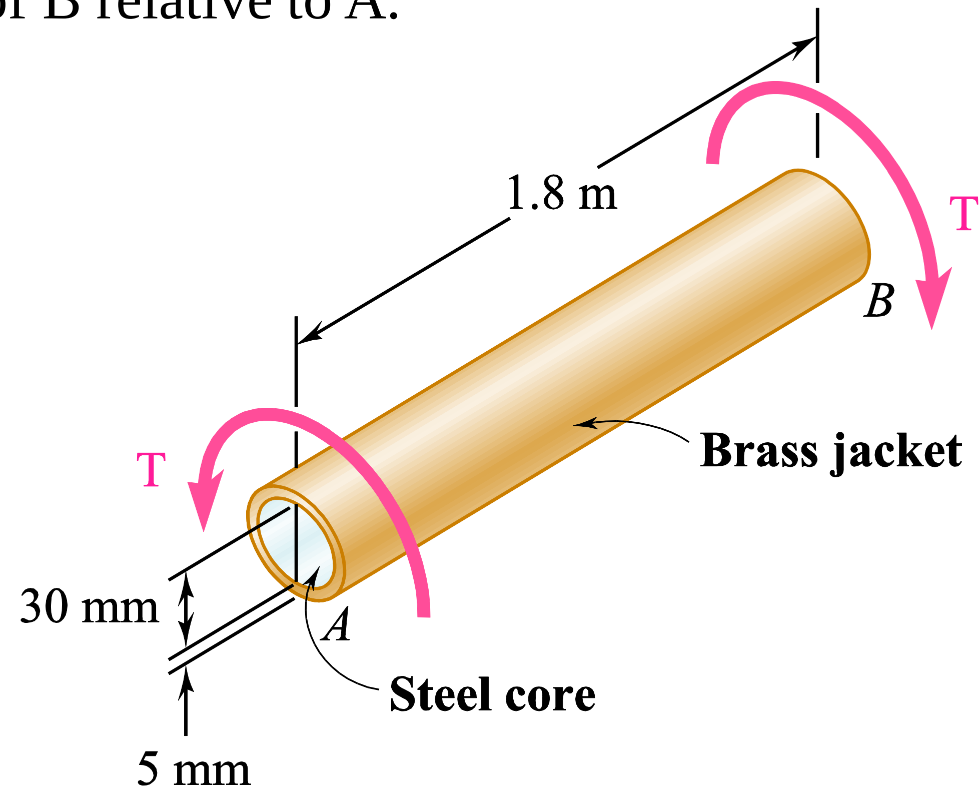
3-51: The solid cylinders AB and BC are bounded together at B and are attached to fixed supports at A and C. Knowing that the modulus of rigidity is 26 GPa for aluminum and 39 GPa for brass, determine the maximum shearing stress **(a)** in cylinder AB, **(b)** in cylinder BC.



Example 8

Example 9

3-53: The composite shaft shown consists of a 5 mm thick brass jacket (G_{brass} = 39 GPa) bonded to a 30 mm diameter steel core (G_{steel} = 77.2 GPa). Knowing that the shaft is subjected to 565 N.m torques, determine **(a)** the maximum shearing stress in the brass jacket, **(b)** the maximum shearing stress in the steel core, **(c)** the angle of twist of B relative to A.



Example 9

Design of Transmission Shafts

Principal transmission shaft performance specifications are: **Power** and **Speed**

Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi f T \quad \Rightarrow \quad T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

Solid shafts

$$\frac{J}{c} = \frac{T}{\tau_{\max}}$$

$$\frac{\pi}{2}c^3 = \frac{T}{\tau_{\max}}$$

Hollow shafts

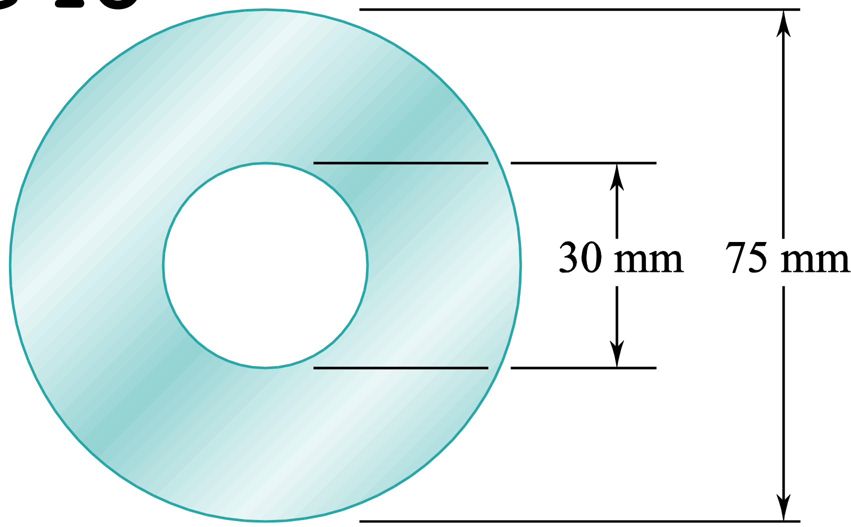
$$\frac{J}{c_2} = \frac{T}{\tau_{\max}}$$

$$\frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}}$$

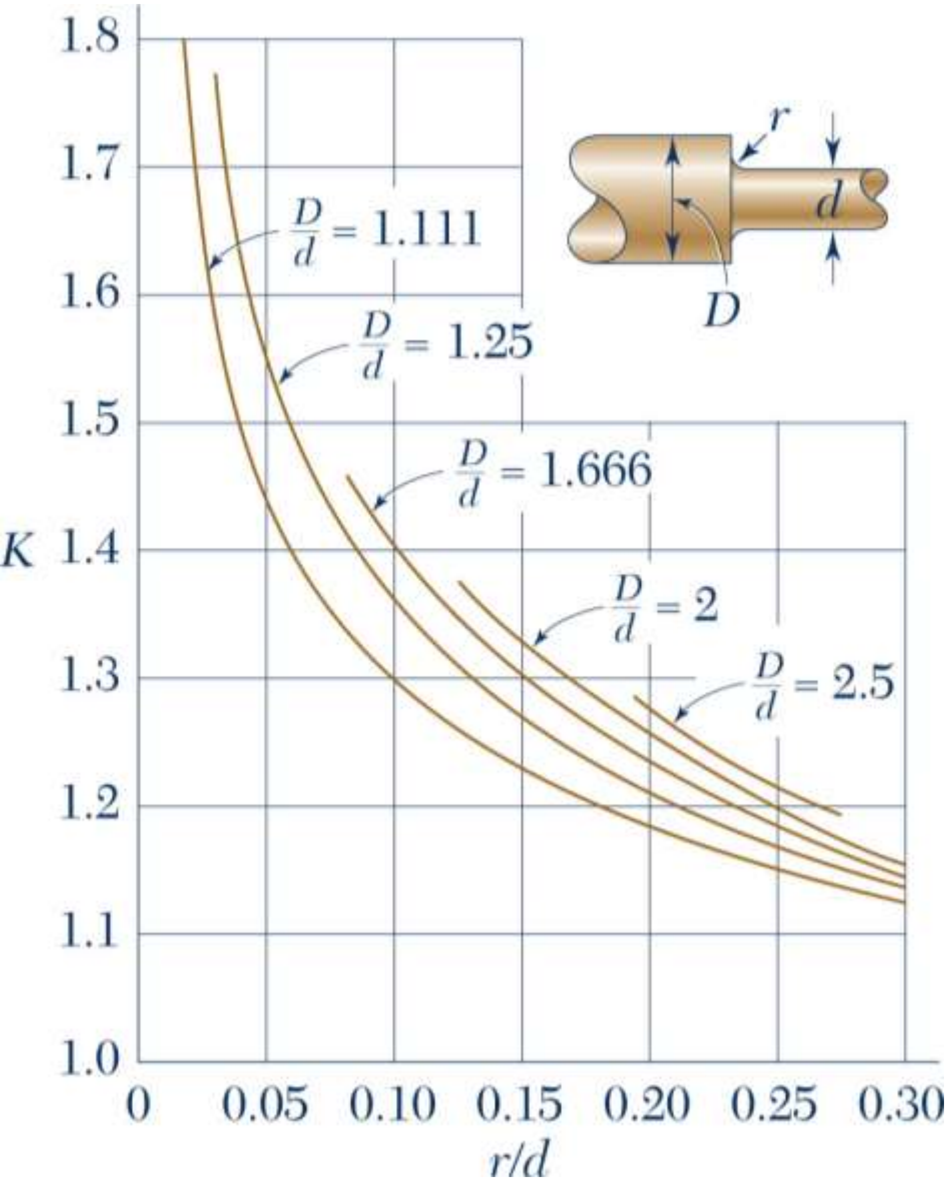
Designer must select shaft material and cross-section to meet performance specifications without exceeding allowable shearing stress.

Example 10

3-68: While a steel shaft of the cross section shown rotates at 120 rpm, a stroboscopic measurement indicates that the angle of twist is 2° in a 4 m length. Using G = 77.2 GPa, determine the power being transmitted.



Stress Concentrations



Stress-concentration factors for fillets in circular shafts.

Assumed a circular shaft with uniform cross-section loaded through rigid end plates. The **maximum ideal shearing stress** is calculated using the formula:

$$\tau_{\max I} = \frac{Tc}{J}$$

Now assume another shaft with cross-sectional discontinuities. This can cause stress concentrations and can raise the stress. Let **K** be the **stress concentration factor**, then the **true maximum stress** can be calculated from the ideal maximum stress as follows:

$$\tau_{\max T} = K\ \tau_{\max I}$$

Example 11

3-87: The stepped shaft shown must rotate at a frequency of 50 Hz. Knowing that the radius of the fillet is r = 8 mm and the allowable shearing stress is 45 MPa, determine the maximum power that can be transmitted.